Introduction to the affine Grassmannian and MV cycles.

Objectives

1. Motivate the study of MV cycles through representation theory.
2. Introduce the correct language and notation.

They give bases for irreps with unique integrality properties (called perfect bases) which help study tensor product mult.

Their existence was known for a while, but they were hard to pinpoint... geometric Satake

§0. An enlightening example.

We begin with the study of $G = Gl_n(\mathbb{C})$ and its fin. dim. representations.

First, $Gl_n(\mathbb{C})$ acts on $\mathbb{C}^n$ through matrix multiplication. Thus, it acts on $\Lambda^k \mathbb{C}^n$ $(1 \le k \le n)$:

$$g \cdot (v_1 \wedge \dots \wedge v_k) = gv_1 \wedge \dots \wedge gv_k \quad \text{(where the action is extended linearly)}$$

Call $\Lambda^k \mathbb{C}^n$ the k^{th} fundamental representation of $Gl_n(\mathbb{C})$.

0.1. Weights

We can decompose $\Lambda^k \mathbb{C}^n$ into so-called weight spaces. Let

$$T = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \in Gl_n(\mathbb{C}) \right\} = \text{diag matrices with } \det \neq 0$$

Then, $T \simeq (\mathbb{C}^\times)^n$ is an abelian group (in fact, maximal such in G). This implies that when one restricts any rep of G to a rep of T, it decomposes into a direct sum of irreps of T, all of which are 1 dim'l since T is abelian. In the above case, consider the basis $e_{i_1} \wedge \dots \wedge e_{i_k} \in \Lambda^k \mathbb{C}^n$, $I = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$, $|I| = k$. Then,

for $t \in T$, $t = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}$ $\quad t \cdot e_{i_1} \wedge \dots \wedge e_{i_k} = te_{i_1} \wedge \dots \wedge te_{i_k} = (t_{i_1} \cdot \dots \cdot t_{i_k})\, e_{i_1} \wedge \dots \wedge e_{i_k}$

$$\Rightarrow \Lambda^k \mathbb{C}^n = \bigoplus_{\substack{I \subseteq \{1, \dots, n\} \\ |I| = k}} \mathbb{C} \cdot e_I \quad \text{is a decomposition of } \Lambda^k \mathbb{C}^n \text{ into irreps of } T.$$

We introduce some terminology. Let $X^* = \mathrm{Hom}(T, \mathbb{C}^\times) \simeq \mathbb{Z}^n$ = weight lattice of (G, T) where

$\hookrightarrow \mathrm{Hom}(\mathbb{C}^\times, \mathbb{C}^\times) \simeq \mathbb{Z}$ (alg. group homs)

$$\left(\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto t_1^{\lambda_1} \cdot \dots \cdot t_n^{\lambda_n} \right) \leftrightarrow (\lambda_1, \dots, \lambda_n)$$

Elements of X^* are called weights. Call $\lambda = (\lambda_1, \dots, \lambda_n)$ dominant if $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_n$ and let $(X^*)_+$ be the set of dominant wts.

We introduce the following partial order on $\mathbb{Z}^n$:

$$(\lambda_1, \dots, \lambda_n) \geq (\mu_1, \dots, \mu_n) \quad \text{if} \quad \lambda_1 \geq \mu_1, \ \lambda_1+\lambda_2 \geq \mu_1+\mu_2, \dots, \ \lambda_1+\dots+\lambda_n \geq \mu_1+\dots+\mu_n.$$

As we saw above, the weight of $\Lambda^k \mathbb{C}^n$ are lists of k 1's.

Ex. $n=4$, $k=2$.

$\dim \Lambda^2 \mathbb{C}^4 = \binom{4}{2} = \frac{4 \cdot 3 \cdot 2}{2 \cdot 2} = 6$

basis	$e_1 \wedge e_2$	$e_1 \wedge e_3$	$e_1 \wedge e_4$	$e_2 \wedge e_3$	$e_2 \wedge e_4$	$e_3 \wedge e_4$
wt	$(1,1,0,0)$	$(1,0,1,0)$	$(1,0,0,1)$	$(0,1,1,0)$	$(0,1,0,1)$	$(0,0,1,1)$
sequences	$1,2,2,2$	$1,1,2,2$	$1,1,1,2$	$0,1,2,2$	$0,1,1,2$	$0,0,1,2$

For the representations $\Lambda^k \mathbb{C}^n$ of $Gl_n(\mathbb{C})$, $e_1 \wedge \dots \wedge e_k$ has weight

$$\varpi_k = (\overbrace{1,1,\dots,1}^{k},\overbrace{0,\dots,0}^{n-k}) = k^{th} \text{ fundamental weight}$$

and every $\lambda \in wt(\Lambda^k \mathbb{C}^n)$ satisfies $\lambda \leq \varpi_k$. The only dominant wt of $\Lambda^k \mathbb{C}^n$ is ϖ_k.

0.2 Irreps

Def. A weight $\lambda \in X^*$ is a highest wt for a rep V if $V_\lambda \neq \{0\}$ and $\mu \leq \lambda$ for all $\mu \in wt(V)$.

Théo

Remark. This definition is not standard! Usually, one defines hw vectors first... Moreover, with the above def, one can show that if λ is a hw of V (a fin. dim rep), then λ is dominant.

If V admits a hw, it is unique by definition.

Thm. For each $\lambda \in (X^*)_+$, there is a unique irrep $V(\lambda)$ of $Gl_n(\mathbb{C})$ having highest weight λ and all irreps occur in this way. One also has $\dim(V(\lambda)_\lambda) = 1$.

One can prove that $\Lambda^k \mathbb{C}^n$ is an irrep. We saw that its hw is ϖ_k.

We show how to construct irreps using tensor products. First, if V, W are G-reps and $v \in V$, $w \in W$, the action of G on $V \otimes W$ is given by $g \cdot v \otimes w = gv \otimes gw$. Consequently, if $v_1, \dots, v_n \in V$ and $w_1, \dots, w_m \in W$ are weight bases, then $v_i \otimes w_j$ is a wt basis of $V \otimes W$ and

$$t \cdot v_i \otimes w_j = tv_i \otimes tw_j = \lambda_i(t)\mu_j(t)\, v_i \otimes w_j = (\lambda_i + \mu_j)(t)\, v_i \otimes w_j$$

and thus, $(V \otimes W)_\mu = \bigoplus_{\mu_1 + \mu_2 = \mu} V_{\mu_1} \otimes W_{\mu_2}$.

It follows that $\left(V(\lambda) \otimes V(\mu)\right)_{\lambda+\mu} = \mathbb{C}\, v_\lambda \otimes v_\mu$ and $\lambda + \mu$ is a hw of $V(\lambda) \otimes V(\mu)$. However, it might not be irreducible

Ex. $n = 3$

let's study $V = V(\varpi_1) \otimes V(\varpi_2) = \overset{\dim 3}{\mathbb{C}^3} \otimes \overset{\dim \binom{3}{2} = 3}{\Lambda^2 \mathbb{C}^3}$. The weights of the two reps are

	$V(\varpi_1)$			$V(\varpi_2)$		
wt basis	e_1	e_2	e_3	$f_1 \wedge f_2$	$f_1 \wedge f_3$	$f_2 \wedge f_3$
weights	$(1,0,0)$	$(0,1,0)$	$(0,0,1)$	$(1,1,0)$	$(1,0,1)$	$(0,1,1)$

$e_1 \otimes f_1 \wedge f_2$ $(2,1,0)$	$e_1 \otimes f_1 \wedge f_3$ $(2,0,1)$	$e_1 \otimes f_2 \wedge f_3$ $(1,1,1)$
$e_2 \otimes f_1 \wedge f_2$ $(1,2,0)$	$e_2 \otimes f_1 \wedge f_3$ $(1,1,1)$	$e_2 \otimes f_2 \wedge f_3$ $(0,2,1)$
$e_3 \otimes f_1 \wedge f_2$ $(1,1,1)$	$e_3 \otimes f_1 \wedge f_3$ $(1,0,2)$	$e_3 \otimes f_2 \wedge f_3$ $(0,1,2)$

We know that V has $V(\varpi_1 + \varpi_2)$ as a factor. However, this rep'n is reducible since one has that $v = e_1 \otimes f_2 \wedge f_3 + e_2 \otimes f_1 \wedge f_3 + e_3 \otimes f_1 \wedge f_2$ is such that $\mathbb{C}v$ is a subrep $\simeq V(\varpi_3)$.

In fact, $V \simeq \underbrace{V(\varpi_1 + \varpi_2)}_{\simeq\, \mathfrak{sl}_3(\mathbb{C}) \text{ with adjoint action}} \oplus V(\varpi_3)$.

$\mathbb{C}^3 \otimes \Lambda^2 \mathbb{C}^3 \simeq \mathbb{C}^3 \otimes (\mathbb{C}^3)^* \simeq \mathrm{End}(\mathbb{C}^3)$

If $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ is dominant, then

$$(\lambda_1 - \lambda_2)\varpi_1 + (\lambda_2 - \lambda_3)\varpi_2 + \dots + (\lambda_{n-1} - \lambda_n)\varpi_{n-1} + \lambda_n \varpi_n$$
$$= \big((\lambda_1-\lambda_2)+(\lambda_2-\lambda_3)+\dots+(\lambda_{n-1}-\lambda_n)+\lambda_n,\ (\lambda_2-\lambda_3)+\dots+(\lambda_{n-1}-\lambda_n)+\lambda_n,\ \dots,\ (\lambda_{n-1}-\lambda_n)+\lambda_n,\ \lambda_n\big) = \lambda$$

and $(\lambda_1 - \lambda_2), (\lambda_2 - \lambda_3), \dots, (\lambda_{n-1} - \lambda_n) \in \mathbb{Z}_{\geq 0}$, $\lambda_n \in \mathbb{Z}$. Conversely, if $m_1, \dots, m_{n-1} \in \mathbb{Z}_{\geq 0}$ and $m_n \in \mathbb{Z}$, $\Sigma m_i \varpi_i$ is dominant.

We deduce that if $\lambda = (\lambda_1, \dots, \lambda_n) \in (X^*)_+$, then if $m_i = \lambda_i - \lambda_{i+1}$ $i = 1, \dots, n-1$, the rep

$$V(\varpi_1)^{\otimes m_1} \otimes \dots \otimes V(\varpi_{n-1})^{\otimes m_{n-1}} \otimes V(\varpi_n)^{\otimes \lambda_n}$$

contains $V(\lambda)$ as a summand. If $\lambda_n < 0$, one needs to tensor with $\big(V(\varpi_n)^*\big)^{-\lambda_n}$.

0.3 Schubert varieties.

$G(k,n)$

First, recall that $G = GL_n(\mathbb{C})$ acts transitively on the set of k-dimensional subspaces of $\mathbb{C}^n$. Also, given $W = \mathrm{span}\{e_1, \dots, e_k\}$, one has that

$$\mathrm{stab}_G(W) = \begin{pmatrix} \boxed{k \times k} & * \\ 0 & \boxed{(n-k) \times (n-k)} \end{pmatrix} = P$$

which shows that $G/P \cong G(k,n)$ via $gP \mapsto \mathrm{span}\{ge_1, \dots, ge_k\}$.

Now, one can also think of $G(k,n)$ as $k \times n$ matrices of rank k

$$gP \mapsto \mathrm{span}\{ge_1, \dots, ge_k\}$$
$$\mathrm{Span}\{v_1, \dots, v_k\} \mapsto \left[\begin{pmatrix} - & v_1 & - \\ & \vdots & \\ - & v_k & - \end{pmatrix}\right]$$

where two matrices A, B are equiv. if there is a $k \times k$ inv. matrix p s.t $pA = B$. This is the action of the parabolic subgroup. But, by elementary lin. alg, any $k \times n$ matrix can be put uniquely via P into row canonical form.

Ex. $n=4$, $k=2$: $Gr(2,4) = \{\text{planes in } \mathbb{C}^4\}$

$$P = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ \multicolumn{2}{c}{\multirow{2}{*}{$\boxed{0}$}} & * & * \\ & & * & * \end{pmatrix} \qquad gP \mapsto \operatorname{span}\{ge_1, ge_2\} \mapsto \begin{pmatrix} g_{11} & g_{21} & g_{31} & g_{41} \\ g_{12} & g_{22} & g_{32} & g_{42} \end{pmatrix}$$

Possible canonical forms of $\left(\overline{}\right)$:

$$\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}, \begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}, \begin{pmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}, \begin{pmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus, given a subset $I \subseteq \{1, \dots, n\}$, $|I| = k$, we can define

$$C_I = \left\{ \begin{pmatrix} - v_1 - \\ \vdots \\ - v_k - \end{pmatrix} ; \begin{array}{l} \text{The matrix reduces} \\ \text{to a r.e.f matrix with 1's at the } i \text{ columns} \end{array} \right\}$$

and $X_I = \overline{C_I}$. It is a fact that $H_\bullet(G(k,n))$ is torsion free and freely generated by the classes $[X_I]$.

End of lecture 1

Upshot: The basis $[X_I]$ coming from geometry relates to the basis e_I of $\Lambda^k \mathbb{C}^n$ in a generalizable way: For $\lambda = \omega_k$, $G = GL_n$ and Gr = affine Grassmannian,

$$Gr^\lambda \simeq G(k,n), \quad IH(Gr^\lambda)_\mathbb{C} \simeq V(\lambda) \qquad IH(Gr^\lambda)_\mathbb{C} \to V_\lambda$$

and MV cycles (subvarieties of Gr^λ) correspond to Schubert varieties $[X_I] \mapsto e_I$.
For more general G, this endows irreps with a combinatorially well behaved basis called the MV basis. The equivalence above will concerne G and $G^\vee$. In the case of GL_n, one has $(GL_n)^\vee \simeq GL_n$ which simplify things greatly.

§1. The affine Grassmannian for GL_n.

based on mostly the first 2 refs

References:

- Achar, perverse sheaves and applications to rep theory, chap 9
- Achar, intro to the affine grassmannians and the geom Satake equiv (lectures), notes on Losev's website
- Richarz, basics on the affine grassmannians
- Zhu, intro to the affine grassmannians and the geom Satake equiv
- Kamnitzer, MV cycles and polytopes

To define this space, we will need the following: let

$$\mathbb{K} := \mathbb{C}((t)) = \text{Laurent series in } t, \quad \mathcal{O} = \mathbb{C}[[t]] = \text{power series in } t.$$

$\nu(fg) = \nu(f) + \nu(g)$

Then, it is known that $\mathbb{K}$ is a field and $\mathcal{O}$ is its ring of integers for the valuation $\nu : \mathbb{K} \to \mathbb{Z} \cup \{\infty\}$ where $\nu(f)$ = degree of the first non-zero coefficient. Then, $\mathcal{O} = \{f \in \mathbb{K} \,;\, \nu(f) \geq 0\}$ making $\mathcal{O}$ a discrete valuation ring ($\Leftrightarrow$ P.I.D. having only one maximal ideal, being $\langle t \rangle$ in this case). One also has $\mathcal{O}^\times = \{a_0 + a_1 t + \dots : a_0 \neq 0\}$ $= \{f \in \mathcal{O} \,;\, \nu(f) = 0\}$ as an application of the Cauchy product formula.

Recall two thms from ring theory:

- If R is a p.i.d., then any finitely generated module over R is isomorphic to

$$\underbrace{R^m}_{\text{free}} \oplus \underbrace{R/(r_1) \oplus \dots \oplus R/(r_n)}_{\text{torsion}}$$

  where $r_1 \mid r_2 \mid \dots \mid r_n$ and $r_i \in R$ are non-invertible, non-zero elements (called invariant factors).
- If R is a d.v.r., then a uniformizer for R is a choice of $\varpi \in R$ s.t. ϖ is irreducible ($\neq 0$, not a unit and not a product of two non-invertible elements). Then, $\langle \varpi \rangle \subseteq R$ is the (unique) max ideal and any ideal is of the form (ϖ^k) for some $k \in \mathbb{Z}_{\geq 1}$.

1.1. Lattices

Fix $n \geq 1$. A lattice $L \subseteq \mathbb{K}^n$ is a free rank n $\mathcal{O}$-submodule, i.e. L admits an $\mathcal{O}$-basis $\ell_1, \dots, \ell_n \in \mathbb{K}^n$ such that for all $\ell \in L$, $\ell = a_1 \ell_1 + \dots + a_n \ell_n$ with $a_1, \dots, a_n \in \mathcal{O}$.

Def (Gr_{Gl_n}). The affine grassmannian for Gl_n is $Gr := \{L : L \text{ is a lattice in } \mathbb{K}^n\}$.

Ex. The standard lattice is $L_0 = \mathcal{O}^n = \text{span}_{\mathcal{O}}\{e_1, \dots, e_n\}$ where $e_i = (1)$ e-pos i.

For $n=2$, one could take

$$L = \text{span}_{\mathcal{O}} \left\{ \begin{pmatrix} t^2+t^3+t^4+\dots \\ t^{-3}+1 \end{pmatrix}, \begin{pmatrix} e^t \\ t^2+2t+1 \end{pmatrix} \right\}$$

Given $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}$, define $L_\lambda = \text{span}_{\mathcal{O}}\{t^{\lambda_1}e_1, \dots, t^{\lambda_n}e_n\}$.

Thus, $L_{\omega_k} = \text{span}_{\mathcal{O}}\{te_1, \dots, te_k, e_{k+1}, \dots, e_n\}$.

The valuation v extends to a valuation on Gr. For L a lattice, let $\ell_1, \dots, \ell_n \in \mathbb{K}^n$ be an $\mathcal{O}$-basis. Then, define

$$v(L) = v\left(\det \begin{pmatrix} | & & | \\ \ell_1 & \cdots & \ell_n \\ | & & | \end{pmatrix} \right)$$

This is well defined since if $g \in Gl_n(\mathbb{K})$, then

$$\det \begin{pmatrix} | & & | \\ g\ell_1 & \cdots & g\ell_n \\ | & & | \end{pmatrix} = \det\left(g \cdot \begin{pmatrix} | & & | \\ \ell_1 & \cdots & \ell_n \\ | & & | \end{pmatrix} \right) = \det(g) \cdot \det \begin{pmatrix} | & & | \\ \ell_1 & \cdots & \ell_n \\ | & & | \end{pmatrix}$$

$$\Rightarrow v\left(\det \begin{pmatrix} | & & | \\ g\ell_1 & \cdots & g\ell_n \\ | & & | \end{pmatrix} \right) = v(\det(g)) + v\left(\det \begin{pmatrix} | & & | \\ \ell_1 & \cdots & \ell_n \\ | & & | \end{pmatrix} \right)$$

Consequently, if $\ell_1', \dots, \ell_n'$ is another basis of L, then the change of basis matrix lies in $Gl_n(\mathcal{O})$ and it is an invertible matrix $\Rightarrow \det(g) \in \mathcal{O}^\times \Rightarrow v(\det g) = 0$.

Ex. $v(L_0) = 1$

$v(L) = v\left((t^2+t^3+\dots)(t^2+2t+1) - (t^{-3}+1)e^t \right) = v(-t^{-3} + \text{higher order terms}) = -3$

as in the previous ex.

$v(L_\lambda) = \lambda_1 + \dots + \lambda_n$.

$v(t^m L_0) = mn$

Lemma. For any lattice L, there are integers $a \leq b$ s.t. $t^b L_0 \subseteq L \subseteq t^a L_0$.

Proof. Let $\ell_i = \sum_{j=1}^n \ell_{ij} e_j$ for $\ell_{ij} \in \mathbb{K}$. Then, let $m = \min\{v(\ell_{ij})\} \in \mathbb{Z}$. It follows that $t^{-m}\ell_1, \dots, t^{-m}\ell_n \in \mathcal{O}^n$
$\Rightarrow t^{-m} L \subseteq L_0 \Rightarrow L \subseteq t^m L_0$.

For the other inclusion, we'll have to work ... (As Antoine L. mentioned, this can be done using dual lattices, see Lusztig "singularities, character formulas, ..." 1983, section 11.)

1.2. Using the orbit-stabilizer thm.

Lemma. There is a transitive action $GL_n(\mathbb{K}) \curvearrowright Gr$ and $\mathrm{stab}(L_0) = GL_n(\mathcal{O})$.

Proof. This is the action we saw above. It is transitive since for any $L = \mathrm{span}_{\mathcal{O}}\{\ell_1, \dots, \ell_n\}$, one has that

$$g = \begin{pmatrix} \top & & \top \\ \ell_1 & \cdots & \ell_n \\ \bot & & \bot \end{pmatrix} \in GL_n(\mathbb{K}) \text{ and } \quad g e_i = \ell_i \Rightarrow g \cdot L_0 = L.$$

Now, if $g \in \mathrm{Stab}(L^0) \Rightarrow g e_i = \sum_{j=1}^n g_{ij} e_j \in \mathcal{O}^n \Rightarrow g_{ij} \in \mathcal{O}\ \forall i,j \Rightarrow g \in GL_n(\mathcal{O})$.
Conversely, if $g \in GL_n(\mathcal{O})$, then $g^{-1} \in GL_n(\mathcal{O})$ and

$$g e_i = \sum g_{ij} e_j \in \mathcal{O}^n \Rightarrow g L_0 \subseteq L_0 \quad \text{and} \quad g^{-1} L_0 \subseteq L_0 \Rightarrow L_0 \subseteq g L_0.$$

Using the orbit-stabilizer thm, this allows us to write $GL_n(K)/GL_n(\mathcal{O}) \overset{1:1}{\leftrightarrow} Gr$ via $g \mapsto g L_0$. → one can endow Gr with a topo using this!

Lemma. $GL_n(\mathbb{K}) = \bigsqcup_{\lambda = (\lambda_1 \geq \dots \geq \lambda_n)} GL_n(\mathcal{O}) t^\lambda GL_n(\mathcal{O})$ where $t^\lambda = \begin{pmatrix} t^{\lambda_1} & & \\ & \ddots & \\ & & t^{\lambda_n} \end{pmatrix}$.

Let $g = (g_{ij}) \in GL_n(\mathbb{K})$. Multiplication by elements of $GL_n(\mathcal{O})$ on the left $\Leftrightarrow$ row operations while mult on the right $\Leftrightarrow$ col operations. Moreover, $GL_n(\mathbb{C}) \subseteq GL_n(\mathcal{O})$.

Step 1. Using permutation matrices (which $\in GL_n(\mathcal{O})$) put the g_{ij} of minimal valuation at the g_{nn} position in g.

Step 2. Let $g_{nn} = t^{\lambda_n} \cdot f$ with $f \in \mathcal{O}^\times$. Multiply by the matrix $\mathrm{diag}(1, \dots, 1, f^{-1}) \in GL_n(\mathcal{O})$ on the left. Then, $g_{nn} = t^{\lambda_n}$ with $\lambda_n \in \mathbb{Z}$ the min valuation of the g_{ij}'s.

Step 3. Clear the n^{th} col $\forall i$: One can write $g_{in} = t^{\lambda_n} \cdot f_i$ and $v(g_{in}) = \lambda_n + v(f_i) \geq v(g_{nn}) = \lambda_n \Rightarrow v(f_i) \geq 0$
$\Rightarrow f_i \in \mathcal{O}$. Multiply by $I_n - f_i E_{in} \in GL_n(\mathcal{O})$ on the left (this clears the (i,n) entry).

↳ $(I_n + \lambda E_{ij})A$: row $i \mapsto$ row $i + \lambda$ row j
row $\ell \mapsto$ row ℓ for $\ell \neq i$

Step 4. Clear the n^{th} row. Same as above, but on the right.
Thus, the matrix now looks like

$$\begin{pmatrix} \boxed{\;*\;} & \begin{matrix} 0 \\ | \\ 0 \end{matrix} \\ 0 \text{ --- } 0 & t^{\lambda_n} \end{pmatrix}$$

and we proceed by induction on n. The resulting matrix is of the desired form.

Corollary. For any lattice L, there exists $\lambda = (\lambda_1 \geq \dots \geq \lambda_n)$ and $g \in GL_n(\mathcal{O})$ s.t. $L = gL_\lambda$.

For $\lambda = (\lambda_1 \geq \dots \geq \lambda_n)$, we define $Gr_\lambda := GL_n(\mathcal{O})L_\lambda$ and $Gr = \bigsqcup_\lambda Gr_\lambda$.

To finish the proof of the lemma from section 1.1., for $\lambda = (\lambda_1 \geq \dots \geq \lambda_n)$, let $m = \lambda_1$. Then, for all i,
$t^m e_i = \underbrace{t^{m-\lambda_i}}_{\in \mathcal{O}} \cdot t^{\lambda_i} e_i \Rightarrow t^m e_i \in L_\lambda \Rightarrow t^m L_0 \subseteq L_\lambda$. Consequently, if L is any lattice, write

$$L = gL_\lambda \text{ for } g \in GL_n(\mathcal{O}).$$

then, $t^m L_0 \subseteq L_\lambda \Rightarrow t^m L_0 = t^m g L_0 \subseteq gL_\lambda = L$ (since $g \in \mathrm{Stab}(L_0)$).

1.3. Relationship with the usual grassmannian.

Lemma. For $L' \subseteq L$ two lattices, L/L' is a f.d. $\mathbb{C}$-vector space of dim $\nu(L') - \nu(L)$.

Proof. Let $g \in GL_n(\mathbb{K})$ s.t. $gL = L_0$. Thus, $gL' \subseteq gL = L_0$. Now, let $h \in GL_n(\mathcal{O})$ s.t. $h(gL') = L_\lambda$. It follows that $L_\lambda = h(gL') \subseteq hL_0 = L_0$. Since $L_\lambda \subseteq L_0$, it follows that $t^{\lambda_i} e_i \in L_0 \Rightarrow \lambda_i \geq 0 \ \forall i$.

Consider L_0/L_λ. This is a f.d. $\mathbb{C}$-vector space having basis

$$e_1, t^1 e_1, \dots, t^{\lambda_1 - 1} e_1, \ e_2, te_2, \dots, t^{\lambda_2 - 1} e_2, \dots \text{ etc}$$

$\Rightarrow \dim (L^0/L_\lambda) = \lambda_1 + \lambda_2 + \dots + \lambda_n$.

But, $L/L' \simeq L_0/L_\lambda$ via hg and

$$\nu(L') - \nu(L) = (\nu(L') + \nu(\det hg)) - (\nu(L) + \nu(\det hg)) = \nu(hgL') - \nu(hgL) = \nu(L_\lambda) - \nu(L_0) = \lambda_1 + \dots + \lambda_n.$$

End of lecture 2.

For $\nu, a, b \in \mathbb{Z}$, $a \leq b$, define

$$Gr^{\nu,[a,b]} := \{ L \in Gr \,;\, \nu(L) = \nu \text{ and } t^b L_0 \subseteq L \subseteq t^a L_0 \}$$ sometimes empty

As we discussed, $\bigcup_{\nu,a,b} Gr^{\nu,[a,b]} = Gr$. Moreover, if $L \in Gr^{\nu,[a,b]}$, then $L/t^b L_0$ is a subspace of dim $nb-\nu$ of $t^a L^0 / t^b L^0$ which has dim $nb - na = n(b-a)$.
This subspace is also t stable, meaning that $t \cdot t^a L_0/L \subseteq t^a L_0/L$, since it is an $\mathcal{O}$-module.
By the 3rd iso thm,

$$\begin{array}{ccc} \mathcal{O}\text{-modules } L \text{ s.t. } t^b L_0 \subseteq L \subseteq t^a L_0 & \overset{1:1}{\leftrightarrow} & \mathcal{O}\text{-submodules } \Lambda \subseteq t^a L_0 / t^b L_0 \\ \nu(L) = \nu & \longleftrightarrow & \dim_{\mathbb{C}} \Lambda = nb - \nu \end{array}$$

which embeds $Gr^{\nu,[a,b]} \hookrightarrow G(nb-\nu, n(b-a))$. one can endow Gr with a topo using this!
If one looks back at the lemma saying that for every L, one can find $a \leq b$ s.t. $t^b L_0 \subseteq L \subseteq t^a L_0$ one finds that $b = \lambda_1$ and $a = \lambda_n$ work, where $L \in Gl_n(\mathcal{O}) \cdot L_\lambda$.
Consequently, for $\lambda = \varpi_k$, we have $b = 1$ and $a = \begin{cases} 0 & k \neq n \\ 1 & k = n \end{cases}$. Also, $\nu(L_{\varpi_k}) = k$. For $k \neq n$, we thus have that $t L_0 \subseteq L \subseteq L_0$ with $\nu(L) = k$. Conversely, if L is such that $t L_0 \subseteq L \subseteq L_0$ then $L \in Gr_\lambda$ with $\lambda_i = 0$ or 1. Putting $\nu(L) = k$ fixes $\lambda = \varpi_k$. We have that

$$Gr_{\varpi_k} \simeq Gr^{k,[0,1]} \overset{*}{\hookrightarrow} G(n \cdot 1 - k, n(1-0)) = G(n-k, n)$$

Lastly, given any L (not necessarly an $\mathcal{O}$-module) with $t L_0 \subseteq L \subseteq L_0$, it follows that

$$t L \subseteq t L_0 \subseteq L \Rightarrow \text{it is an } \mathcal{O}\text{-module}$$

which shows that the inclusion $*$ is an equality : $Gr_{\varpi_k} \simeq G(n-k, n)$

1.4 Semi-infinite orbits and MV cycles

Using the same Gaussian elimination trick, on can show that $G(\mathbb{K}) = \bigsqcup_{\lambda \in X^*} N(\mathbb{K}) t^\lambda Gl_n(\mathcal{O})$ where

Iwasawa decomposition

$$N = \begin{pmatrix} 1 & & * \\ & 1 & \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \subseteq Gl_n$$

Def. The positive/negative semi-infinite orbits are (respectively) the subsets

$$S_+^\mu = N(\mathbb{K})t^\mu \qquad S_-^\mu = N_-(\mathbb{K})t^\mu$$

of Gr.

MV cycles are irreducible components of $\overline{S_+^\mu \cap \mathrm{Gr}_\lambda}$.

To do: give an explicit description of S_+^μ in the lattice model and show that the $S_+^\mu \cap \mathrm{Gr}_\lambda$ correspond to Schubert varieties in the case where $\lambda = \varpi_k$.

§ 1.5 The affine grassmannian of SL_n.

For SL_n, one still has the double coset decomposition

$$SL_n(\mathbb{K}) = \bigsqcup_{\substack{\lambda_1 \geq \dots \geq \lambda_n \\ \lambda_1 + \dots + \lambda_n = 0}} SL_n(\mathcal{O}) t^\lambda SL_n(\mathcal{O})$$

and if one define $\mathrm{Gr}_{SL_n} := SL_n(\mathbb{K})/SL_n(\mathcal{O})$, the embedding $SL_n \hookrightarrow GL_n$ defines an embedding $\mathrm{Gr}_{SL_n} \hookrightarrow \mathrm{Gr}_{GL_n}$. We show that

$$\mathrm{Gr}_{SL_n} \xrightarrow{\sim} \mathrm{Gr}_{GL_n}^{\nu=0} := \{L \in \mathrm{Gr}_{GL_n} ; \nu(L) = 0\}$$

The map is clearly injective. Now, if $L = gL_\lambda \in \mathrm{Gr}_{GL_n}^{\nu=0}$, then $\det(g) \in \mathcal{O}^\times$ and the matrix

$$h = \begin{pmatrix} (\det g)^{-1} & & 0 \\ & 1 & \\ 0 & & \ddots \ 1 \end{pmatrix} \in GL_n(\mathcal{O}) \text{ has } \det h \in \mathcal{O}^\times.$$

It follows that $ht^\lambda = t^\lambda h \Rightarrow hL_\lambda = L_\lambda$ and $gh \in SL_n(\mathcal{O}) \Rightarrow L = ghL_\lambda$

$\Rightarrow$ the map $\mathrm{Gr}_{SL_n} \to \mathrm{Gr}_{GL_n}$ is surjective and injective.

Joel said: "This weird trick works since the ind-scheme Gr_{GL_n} is not reduced".

§2 Notation (2.1 in MV cycles and polytopes)

General notion

- G = connected, simply connected, semisimple complex group
- T = maximal torus of G
- $X^* = \mathrm{Hom}(T, \mathbb{C}^\times)$ = weight lattice
- $X_* = \mathrm{Hom}(\mathbb{C}^\times, T)$ = coweight lattice
- B = Borel subgroup of G containing T ↳ maximal closed connected solvable subgrp
- $\mathfrak{g} = \mathrm{Lie}(G)$

Example in type A.

- $G = SL_n(\mathbb{C})$
- $T = \{\mathrm{diag}(t_1, \dots, t_n)\ ;\ t_i \in \mathbb{C}^\times,\ \prod t_i = 1\}$
- $X^* \simeq \{(\mu_1, \dots, \mu_n) \in \mathbb{Z}^n\} / \mathbb{Z}(1,1,\dots,1)$
- $X_* \simeq \{(\mu_1^\vee, \dots, \mu_n^\vee) \in \mathbb{Z}^n\ ;\ \mu_1^\vee + \dots + \mu_n^\vee = 0\}$
- $B = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$
- $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}) = \{X \in M_{n\times n}(\mathbb{C})\ ;\ \mathrm{tr}(X) = 0\}$

Let us now define roots.

G acts on $\mathfrak{g} = \mathrm{Lie}(\mathfrak{g})$ via the adjoint representation Ad_G. When $G = SL_n$, $\mathrm{Ad}(g)X = gXg^{-1}$. One can see that $\mathfrak{g}$ is a weight module for G. One has that if E_{ij} is the (i,j)-elementary matrix,

$$t \cdot E_{ij} = \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix} E_{ij} \begin{pmatrix} t_1^{-1} & & & \\ & t_2^{-1} & & \\ & & \ddots & \\ & & & t_n^{-1} \end{pmatrix} = t_i t_j^{-1} E_{ij} = (e_i - e_j)(t) E_{ij}$$

$\Rightarrow E_{ij}$ is an eigenvector under T of wt $e_i - e_j$ and $\mathfrak{sl}_n = \bigoplus_{i=1,\dots,n-1} \mathbb{C}(E_{ii} - E_{i+1,i+1}) \oplus \bigoplus_{i \neq j} \mathbb{C}E_{ij}$.

- Δ = roots of G = non-zero weight of $\mathfrak{g}$
- $\Delta = \{e_i - e_j : i \neq j\}$

Let us now define coroots.

For each $\alpha \in \Delta$, $\dim \mathfrak{g}_\alpha = 1$ and one can find $E_\alpha \in \mathfrak{g}_\alpha$, $F_\alpha \in \mathfrak{g}_{-\alpha}$ s.t. $E_\alpha, F_\alpha, H_\alpha := [E_\alpha, F_\alpha]$ satisfying the property that

$$\varphi_\alpha : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g} \qquad \varphi_\alpha(X) = X_\alpha \quad X \in \{E, F, H\}$$

is an injective Lie alg. hom. For SL_n, one has $\mathfrak{g}_{e_i - e_j} = \mathbb{C}E_{ij}$, $\mathfrak{g}_{e_j - e_i} = \mathbb{C}E_{ji}$ and $[E_{ij}, E_{ji}] = E_{ii} - E_{jj}$. Then, one has that

$$\varphi_{e_i - e_j} : \mathfrak{sl}_2 \to \mathfrak{g} \qquad E \mapsto E_{ij},\ F \mapsto E_{ji},\ H \mapsto E_{ii} - E_{jj}$$

is an injective morphism.

On the level of groups, there is a map

$\Psi_{e_i-e_j} : Sl_2(\mathbb{C}) \to Sl_n(\mathbb{C})$ $\quad \Psi\begin{pmatrix} a & b \\ c & d\end{pmatrix} = a\, e_{ii} + d e_{jj} + b e_{ij} + c e_{ji} + \sum_{k \neq i,j} e_{kk}$

inducing $\psi_{e_i-e_j}$ via differentiation. On the level of tori, this gives a map $\mathbb{C}^\times \to T$:

$$\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix} \mapsto t e_{ii} + t^{-1} e_{jj} + \sum_{k\neq i,j} e_{kk}.$$

Morphisms $\alpha : \mathbb{C}^\times \to T$ coming from $\Psi_\alpha : Sl_2 \to G$ (which in turn came from $\psi_\alpha : \mathfrak{sl}_2 \to \mathfrak{g}$) are called coroots.

- $\Delta^\vee$ = coroots of G

* $\Delta^\vee = \{ t \in \mathbb{C}^\times \mapsto t e_{ii} - t^{-1} e_{jj} + \sum_{k\neq i,j} e_{kk} \}$.
$= \{ e_i - e_j : i \neq j \}$

There is a natural pairing between weights and coweights:

$$\langle\, ,\, \rangle : X^* \otimes X_* \longrightarrow \mathbb{Z}$$

$\lambda \circ \mu^\vee \in \mathrm{Hom}(\mathbb{C}^\times, \mathbb{C}^\times) \simeq \mathbb{Z} \Rightarrow \lambda(\mu^\vee) = (t \mapsto t^k)$. We declare $\langle \lambda, \mu^\vee \rangle = k$.
For Sl_n, we saw that

$$X^* \simeq \{(\mu_1, \dots, \mu_n) \in \mathbb{Z}^n\} / \mathbb{Z}(1,1,\dots,1) \quad \text{and} \quad X_* \simeq \{(\mu_1^\vee, \dots, \mu_n^\vee) \in \mathbb{Z}^n : \textstyle\sum \mu_i^\vee = 0\}$$

and given $\mu \in X^*$, $\mu^\vee \in X_*$, one has

$$\mu(\mu^\vee) = \mu\begin{pmatrix} t^{\mu_1^\vee} & & \\ & \ddots & \\ & & t^{\mu_n^\vee}\end{pmatrix} = (t^{\mu_1^\vee})^{\mu_1} \cdot \ldots \cdot (t^{\mu_n^\vee})^{\mu_n} = t^{\mu_1\mu_1^\vee + \dots + \mu_n\mu_n^\vee}$$

$$\langle \mu, \mu^\vee \rangle = \mu_1\mu_1^\vee + \dots + \mu_n\mu_n^\vee.$$

- W = Weyl group of $(G,T) = N_G(T)/T$
  $N_G(T)$ = Normalizer of T in G

* $W = \mathfrak{S}_n$ where $w \in W$ is uniquely rep by a permutation matrix.

We now define the notion of positive roots. We give two equivalent definitions.
Given a choice of Borel subgroup $B \subseteq G$, one has $\mathrm{Lie}(B) \subseteq \mathfrak{g}$ and declare $\alpha \in \Delta$

positive if $\mathfrak{g}_\alpha \subseteq \mathfrak{b}$. For Sl_n, we fixed $\qquad \mathfrak{g}_{e_i-e_j} = \mathbb{C}E_{ij}$

$$B = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \Rightarrow Lie(B) = \left\{\begin{matrix}\text{upper triang.}\\ \text{traceless}\end{matrix}\right\} \Rightarrow \Delta^+ = \{e_i - e_j : i<j\}$$

Equivalently, for $\mu^\vee \in X_*$ s.t. $\langle \alpha, \mu^\vee \rangle \neq 0 \ \forall \alpha \in \Delta$ one can declare

$$\Delta^+(\mu^\vee) = \{\alpha \in \Delta ; \langle \alpha, \mu^\vee \rangle > 0\}.$$

For Sl_n, choose $\mu^\vee = (n-1, n-2, \dots, -(n-2), -(n-1))$ and thus,

$$\langle e_i - e_j, \mu^\vee \rangle = 2(j-i) \neq 0 \quad \text{and} \quad \Delta^+(\mu^\vee) = \{e_i - e_j ; i<j\}$$

as above.

In a nutshell: $\quad$ choice of positive roots $\Delta^+ \subseteq \Delta$ $\overset{1:1}{\longleftrightarrow}$ Choice of Borel subgrp $T \subseteq B \subseteq G$

End of lecture 3 (and end of lectures!)

We define simple roots. Given Δ^+, let $\Pi = \{\alpha \in \Delta^+ ; \alpha$ cannot be written as a sum of two positive roots$\}$. For Sl_n, $(e_i - e_j) + (e_k - e_l)$ is a root iff $i = l$ or $k = j$. Thus, if $i<j$ and $k<l$, wlog, take $i<j\leq k<l$. Then, $e_i - e_j$ simple $\Leftrightarrow$ There is no k s.t. $(e_i - e_k) + (e_k - e_j) = e_i - e_j$ with $i<k<j$ $\Leftrightarrow$ $j = i+1$.

$$\Rightarrow \Pi = \{e_i - e_{i+1} ; i = 1, \dots, n-1\}$$

Positive and simple coroots are defined analogously.

- N = unipotent radical of B $\qquad\qquad * \ N = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}$
  = subgroup of unipotent elements of the radical of B
  (unipotent: $(g-1)$ nilpotent; radical: maximal normal solvable subgrp)

- $\varpi_1, \dots, \varpi_r \in X^*$ fundamental weights are defined as satisfying $\qquad * \ \varpi_k = (\underbrace{1, 1, \dots, 1}_{k}, 0, 0, \dots, 0) \bmod (1, \dots, 1)$

$$\langle \varpi_i, \alpha_j^\vee \rangle = \begin{cases} 1 & i=j \\ 0 & \text{otherwise} \end{cases}$$

- $a_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle$

  ↳ Amazingly, for general $\mathfrak{g}$, one has that $a_{ij} \in \{0, 2, -1, -2, -3\}$ and $\mathfrak{g}$ is called simply laced if $a_{ij} \neq -2, -3$ $\forall i, j$.

★ For $\mathfrak{sl}_n$, $a_{ij} = 2, -1$ or 0.

- Cartan matrix $= C = (a_{ij})$

★ $$C_{\mathfrak{sl}_n} = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & 2 & \ddots & \\ & \ddots & 2 & -1 \\ 0 & & -1 & 2 \end{pmatrix}$$

- Weyl vector $\rho = \frac{1}{2}\left(\sum_{\alpha \in \Delta^+} \alpha\right) \in X^*$

  It satisfies
  $$\rho = \sum \varpi_i \text{ and } \langle \rho, \alpha_i^\vee \rangle = 1 \ \forall i.$$

★ $\rho = \frac{1}{2}\left(\sum_{i<j} e_i - e_j\right) = \frac{1}{2}(n-1, n-3, \dots, -(n-3), -(n-1))$

- Γ = Dynkin diagram of $\mathfrak{g}$

  Vertices index the set Π and i and j are linked by $|a_{ij}|$ edges for $i \neq j$.

  (This justifies the terminology "simply laced")

★ $\Gamma_{\mathfrak{sl}_n}$: $\underset{1}{\bullet} - \underset{2}{\bullet} - \underset{3}{\bullet} - \cdots - \underset{n-1}{\bullet}$